Applications of the Boltzmann Equation I.

→ Chapman - Enskog ⇝ Basics (C-E Light)

- a major application of B.E. is calculation of transport coeffs.

- recall, fluid equations [redacted] involve momentum, heat flux

i.e. really $\frac{\partial \underline{v}}{\partial t} = -\nabla \cdot \underline{\underline{\Pi}}$, etc.

continuity form

here:

$$\Pi_{\alpha,\beta} = mn\left(v_\alpha V_\beta + \langle v'_\alpha v'_\beta \rangle\right)$$

$$\langle v'_\alpha v'_\beta \rangle = \int d^3v \; f \, v'_\alpha v'_\beta$$

if $f = f_0 = \frac{n(x)}{(2\pi)^{3/2} v_{Th}(x)^3} \exp\left[-\frac{(\underline{v}-\underline{V}(x))^2}{2 v_{Th}^2}\right]$

⇒

$$\langle v'_\alpha v'_\beta \rangle = \frac{1}{3} \langle v^2 \rangle \delta_{\alpha,\beta}$$

$$\langle v^2 \rangle = 3T/m$$

but [redacted] is $f = f_0$??

recall, f satisfies:

f_0 is soln. to $C(f)=0$.

$$\partial_t f + \underline{v}\cdot\underline{\nabla} f = C(f)$$

$\Rightarrow$ f_0 cannot solve B.E. unless $\nabla f_0 = 0$

can assign time scales:

$\partial_t f \to \omega$

$\underline{v}\cdot\underline{\nabla} f \Rightarrow v_{th}/L$

$C(f) \to \nu$; $\nu = \dfrac{v_{th}}{l_{mfp}}$

$\left\{\begin{array}{l}\text{collisional regime} \\ \text{has } \nu > \dfrac{v_{th}}{L}, \omega\end{array}\right.$

$\rightsquigarrow$ f_0 is 0^{th} order solution.

$l_{mfp} = 1/n\sigma$

then if f_0 inhomogeneous $\Rightarrow$ stationary solution has correction

$f = f_0 + \delta f$

$\hookrightarrow$ $\sim$ inhomogeneity ! $\to$ more precisely response to inhomogeneity

i.e. ∇T, ∇V

and $\langle v'_\alpha v'_\beta\rangle = \int d^3v\,(f_0 + \delta f)\, v'_\alpha v'_\beta$

$= p\,\delta_{\alpha,\beta}$ + viscous stress viscosity $\sim mn\,D$

$D \sim l_{mfp} V_{th}$

i.e. $$mn\langle V'_\alpha V'_\beta\rangle_{visc} = -\eta\,\frac{\partial\langle V_\alpha\rangle}{\partial x_\beta} + \cdots$$

↳ generic form of viscous stress

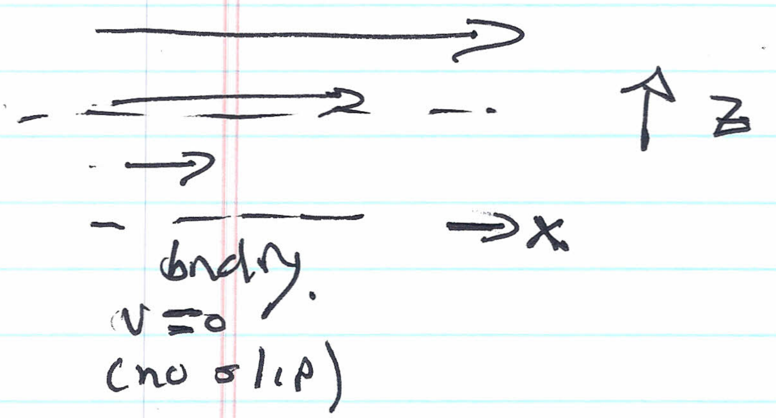
so, need:

- understand viscosity, etc.
- see how calculate δf and then transport coeffs (viscosity)

so what is viscosity about?

- simple physics of transport coeffs:

Consider collisional gas:

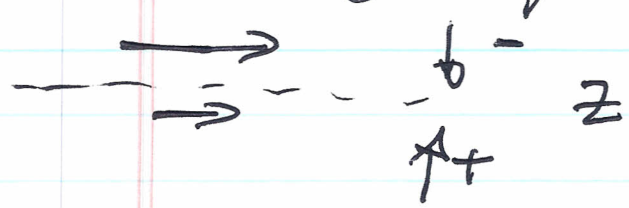


mean flow

$\underline{V} = V(z)\,\hat{x}$

i.e. shear flow (gradient ⊥ flow)

choose imaginary surface



Calculate transport of $\hat{x}$ momentum thru surface:

$$\Pi_+ = \int_{v_z>0} d\underline{v}\; m V_x v_z f$$

$$\sim v_{Th} \bar{V} n m_+$$

$$\Pi_- = \int_{V_z<0} d\underline{v}\; m \cdot V_x V_z f$$

at first glance, would appear $\Pi_+ = \Pi_-$

so $\Pi_{tot} = 0$ but:

 ℓ_{mfp}

→ "scale of resolution" for imaginary surface is ℓ_{mfp} ⇒ defines effective thickness.

→ $V(z)$ has gradient across this.

so

$$\Pi_{tot} = \Pi_- + \Pi_+$$

So

$$\Pi \cong \overbrace{-nm v_{Th} \bar{V}\left(z + \frac{l_{mfp}}{2}\right)}^{\Pi_-}$$

$$\underbrace{+ nm v_{Th} \bar{V}\left(z - \frac{l_{mfp}}{2}\right)}_{\Pi_+}$$

$$\cong -nm \, v_{Th} \, l_{mfp} \frac{\partial V}{\partial z}$$

$$\sim l_{mfp} v_{Th}$$

$$\cong -nm \, D \, \frac{\partial V}{\partial z}$$

$$= -\eta \frac{\partial V}{\partial z}$$

viscosity (shear)

$\sim \rho D$

→ Key Points:

- equal # collisional kicks in +, − direction, but
- more momentum kicked down from above, the velocity gradient.

⇒

- net viscous momentum transport via collisions to relax gradient

How calculate systematically?

$\Rightarrow$ Chapman - Enskog expansion!

now

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\underline{\nabla} f = C(f)$$

$\frac{\partial f}{\partial t}$ — multiple time scales

$\underline{v}\cdot\underline{\nabla} f \sim \frac{v_{Th}}{L}$, $\quad C(f) \sim \nu_{eff.}$

and seek:

f norm ton

$$\Pi_{z,x} = \int d^3v\, v_z\,(m v_x f)$$

$\Pi_{z,x}$: z direction flux of $\hat{x}$ momentum

$$f = f_0 + \delta f$$

$$\Pi_{z,x} = \int d^3v\, v_z\,(m v_x (f_0 + \delta f))$$

if
$$f_0 \cong \frac{n_0}{v_{Th}^3} \exp\left[-\frac{(\underline{v} - V(z)\hat{x})^2}{2 v_{Th}(x)^2}\right]$$

(i.e. local Maxwellian)

→ f_0 contribution vanishes by symmetry!

so

$$\Pi_{z,x} = \int d^3v \text{ [redacted] } v_z \left(m \overset{v_x}{\text{[redacted]}}\right) \delta f$$

drives the flux.

How get δf?

⇒ Perturbative solution!?

$\underline{v} \cdot \nabla f = C(f)$ [redacted] → really an integral equation!

l. O.: $C(f) = 0$

$f = f_0$ → Local Maxwellian

1st O.:

$\underline{v} \cdot \underline{\nabla} f_0 = C(\delta f)$

$\therefore \quad \delta f = C^{-1}\left[\underline{v} \cdot \underline{\nabla} f_0\right]$

How?

→ lengthy calculation (coming)

→ Krook (crook) Model

$$C(f) = -\nu(f - f_0)$$

collisional decay to local Maxwellian

caveat Emptor!

or

$$\underline{v} \cdot \underline{\nabla} f = C(f) = -\nu(f - f_0)$$

$$f = f_0 + \delta f$$

$$\underline{v} \cdot \underline{\nabla}(f_0 + \delta f) = -\nu(f - f_0)$$

l.o. $\quad -\nu(f - f_0) = 0$

$$f = f_0$$

1st order

$$\underline{v} \cdot \underline{\nabla} f_0 = -\nu\,\delta f$$

$$\delta f = -\frac{\underline{v} \cdot \underline{\nabla}}{\nu} f_0$$

perturbative correction to f_0; $\quad O\left(\frac{v_{Th}}{L\nu}\right)$

So

$$\Pi_{z,x} = \int d^3v \; v_z \, m v_x \, \delta f$$

$$= \int d^3v \; v_z n m v_x \left(- \frac{v_z}{\nu} \frac{\partial}{\partial z} f_0 \right)$$

$$\vec{V} \hat{x} = V(z) \hat{x}$$

now

$$f_0 \cong \frac{n(\underline{x})}{v_{Th}^3(\underline{x})} \exp\left[- \frac{(v - V(z)\,\hat{x})^2}{2 v_{Th}^2(\underline{x})} \right]$$

$$= \frac{n}{v_{Th}^3} \exp\left[- \frac{(v - V(z))^2}{2 v_{Th}^2} \right]$$

$$\frac{\partial f_0}{\partial z} = \frac{n}{v_{Th}^3} \frac{v_x}{v_{Th}^2} \frac{\partial V(z)}{\partial z} \exp[\;\;]$$

$$- 2 \frac{V(z)}{v_{Th}^2} \cancel{\frac{\partial V(z)}{\partial z}} \exp[\;\;]$$

dropped ⇒ interested in only linear response of flux to gradient.

see 9a

⇒

$$\frac{\partial f_0}{\partial z} = f_0 \frac{v_x}{v_{Th}^2} \frac{\partial V(z)}{\partial z}$$

Note:

→ here seek linear relation between flux and gradient

→ presumes weak distortion from Maxwellian, i.e.

$$f_0 = \frac{n}{v_{Th}^3} \exp\left[- \frac{(v - V(x)\hat{x})^2}{2 v_{Th}^2} \right]$$

$$= \frac{n}{v_{Th}^3} \exp\left[- \frac{(v^2 - 2vV + V^2)}{2 v_{Th}^2} \right]$$

$$\cong \frac{n}{v_{Th}} \exp\left[\frac{-v^2}{2 v_{Th}^2} \right] \left[1 + \frac{v_x V}{v_{Th}^2} \right]$$

$$= f_0 \left(1 + \frac{v_x V}{v_{Th}^2} \right)$$

e.o. factor on expansion

yields result.

so

$$\Pi_{z,x} = \int d^3x \; v_z \, m v_x \left(-\frac{v_z}{\nu} \frac{v_x}{v_{Th}^2} \right)$$

$$\quad \times f_0 \frac{\partial V(z)}{\partial z}$$

$$= -\# \frac{mn}{\nu} v_{Th}^2 \frac{\partial V(z)}{\partial z}$$

$$= -\# \, mn \underbrace{\left(\frac{v_{Th}}{\nu}\right)}_{l_{mfp}} v_{Th} \frac{\partial V(z)}{\partial z}$$

$$D \equiv v_{Th} \, l_{mfp}$$

$$\boxed{\begin{aligned} \Pi_{z,x} &= -\# \, mn D \frac{\partial V(z)}{\partial z} \\ &= -\# \rho D \frac{\partial V(z)}{\partial z} \end{aligned}}$$

$$\eta = -\# \rho D$$

⇒ basic result for collisional viscosity!

Note form of result:

$$\underset{\text{flux}}{\Pi_{z,x}} = - \underbrace{\#nm\lambda}_{\text{microscopics}} \frac{\partial V_x(z)}{\partial z}$$

flux

microscopics → transport coefficient

→ gradient of thermo. quantity (dist.) → thermodynamic force.

⇒ example of thermodynamic flux - force relation.

⇒ constitutive relation, proportionality is transport coefficient

In general, have vector relation:

$$\underline{\Gamma} = - \underline{\underline{K}} \cdot \underline{\nabla} C$$

vector of fluxes

vector of gradients

matrix of transport coefficients - Onsager matrix)

(n.b. "Onsager symmetry" → $\underline{\underline{K}}$ symmetric)

$\Rightarrow$ Observe:

- $f_{\substack{Max\\eq}}$ annihilates collision operator, and corresponds to $\frac{dS}{dt} = 0$ state $\Rightarrow$ maximal entropy.

but

- $f_{Max}[n(\underline{x}), T(\underline{x}), V(\underline{x})]$ does not satisfy Boltzmann eqn. $\Rightarrow \delta f$ needed.

i.e. $\underline{v} \cdot \underline{\nabla} f = C(f)$

$$\Rightarrow \quad f = f_{Max} + \delta f$$

why: gradients in thermodynamic quantities $\Rightarrow$ system is not in maximum entropy state i.e. $\delta f \neq 0$

- so $\delta f \sim \nabla C$, $\Rightarrow$ relaxation to maximum entropy state will occur by collisional transport

- can describe relaxation macroscopically, i.e.

$$\underbrace{\Pi_{\alpha,\beta}}_{\text{flux}} = -\rho D \underbrace{\frac{\partial \bar{V}_\beta}{\partial x_\alpha}}_{\text{force}}$$

$$-\frac{\partial V_\beta}{\partial x_\alpha} \Pi_{\alpha,\beta} = \rho D \left(\frac{\partial \bar{V}_\beta}{\partial x_\alpha}\right)^2$$

$$\Rightarrow \boxed{\frac{dS}{dt} = \frac{\rho D}{T}\left(\frac{\partial V_\beta}{\partial x_\alpha}\right)^2}$$

— entropy production due transport-induced relaxation

Note time scales:

i.) to form local Maxwellian, H-thm. $\Rightarrow \tau_{coll} \sim \nu^{-1}$

ii.) to form global maximum entropy state:

$$1/\tau_{relax} \sim D/L_V^2 \sim \nu \frac{l_{mfp}^2}{L_V^2}$$

$$L_V^{-1} = \frac{1}{V}\frac{\partial V}{\partial x}$$

$$\tau_{relax} \sim \left(L_v / l_{mfp}\right)^2 \tau_{coll}.$$

$\Rightarrow$ entropy production / relaxation is multiple time scale process!.

More generally, can write:

$$\underbrace{J_i}_{i\text{th flux}} = - \sum_{j=1}^{n} \underbrace{\alpha_{ij}}_{\text{kinetic coefficient}} \; X_j \Rightarrow \text{driving force}$$

$$\frac{dS}{dt} = \varphi = \text{"Dissipation Function"}$$

$$\Rightarrow \quad \frac{dS}{dt} = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{i,j} X_i X_j$$

$\underline{\text{i.e.}}$ $$\frac{dS}{dt} = - X_j J_i$$

so, for 2x2:

$$\frac{dS}{dt} = \sum_{i=1}^{2} \sum_{j=1}^{2} \alpha_{1,1} X_1^2 + (\alpha_{1,2} + \alpha_{2,1}) X_1 X_2 + \alpha_{2,2} X_2^2$$

clearly need:

$$\alpha_{1,1} \geq 0 \qquad \rightarrow \text{ i.e. diffusion}$$
$$\alpha_{2,2} \geq 0 \qquad \text{down gradient}$$

and $$\alpha_{1,1}\,\alpha_{2,2} - \frac{1}{4}\left(\alpha_{1,2} + \alpha_{2,1}\right)^2 \geq 0.$$

(i.e. descr $\Rightarrow$ diag.)

N. B. Off-diagonals ?!

i.e. Can ∇T drive a density flux?

$$\Gamma = \int d^3v\; \underline{v}\; \delta f$$

$$\delta f = -\frac{1}{\nu}\, \underline{v} \cdot \nabla f_0$$

$$\Rightarrow \Gamma = \int d^3v\; \underline{v} \left(-\frac{1}{\nu} V_x \frac{\partial}{\partial x} \left(\frac{n_0}{V_{th}(x)^3} \exp\left[\frac{-mV^2}{2T(x)} \right] \right) \right)$$

clearly can get contribution to Γ_x.

$\rightarrow$ Thermal diffusion.

Message: Gradient in distribution function is the key! ... but integration symmetries matter.